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Nonlinear asymptotic short-wave models in fluid dynamics

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Abstract

Nonlinear monochromatic short surface waves in ideal fluids are studied and, by the general consideration of wave dynamics and perturbative methods a simple and effective multiscale approach is devised for nonlinear asymptotic short-wave dynamics in dispersive systems. In particular the evolution of a monochromatic surface wave in an ideal fluid is shown to lead to a modified Green–Naghdi system of equations and a Green–Naghdi system with surface tension. Short surface waves exist in these systems and the nonlinear asymptotic analysis produces the nonlinear model equations that govern their dynamics. Particular solutions are shown. Moreover the method allows for a general classification of classical model equations as Boussinesq, Benjamin–Bona–Mahony–Peregrine and Camassa–Holm equations. Their related nonlinear short-wave model limits are then derived. A relation between the short-wave limit of the integrable Camassa–Holm equation and the Harry–Dym hierarchy is finally unveiled.

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1. Introduction: long- and short-wave dynamics

The propagation of waves in an ideal incompressible fluid is a classical subject of investigation in mathematical physics. Particular surface gravity waves have been intensively studied and many model equations have been introduced to handle this problem. The main motivation is the fact that the initial three-dimensional water wave problem is analytically intractable and thus, since the classical works of Boussinesq [1] and Korteweg and de Vries [2] a great deal of effort has been made to establish approximate theories.

Great simplifications are obtained by considering the nonlinear evolution of monochromatic long waves with small amplitude in shallow fluids (*long waves in shallow fluids*). It has produced several long-wave model equations among which are the different versions of the Boussinesq equations [3], the Korteweg–de Vries (KdV) equation [4], the modified KdV equation [5], the Kadomtsev–Petviashvili equation [6], the Benjamin–Bona–Mahony–Peregrine equation (BBMP) [7], and the more recent Camassa–Holm equation

(CH) [8]. All these long-wave models are evolution equations with nonlinearity and dispersion, a large number of which are *universal* [9]. Most of them are one-dimensional in space and the general idea behind them being that these approximations are satisfactory models appropriately representing the more general and complete three-dimensional problem. Their solutions are nonlinear structures that propagate without dispersion conserving their shape (solitary waves). Some of the models are completely integrable systems [10]. Except for CH the structural stability of solitary waves comes from an equilibrium between nonlinearity and linear dispersion.

These long-wave models are obtained through the use of some perturbative procedure, one of the more powerful being the method of multiple-scales, or the reductive perturbation method [11–13]. A fundamental step in this method is the Gardner–Morikawa transformation [14]. This transformation enables us to introduce slow space and time variables ξ and τ capable of describing the system asymptotically in space and time. The resulting nonlinear evolution equations are written in these variables and represent asymptotic behaviours in time of long monochromatic waves submitted to nonlinearity and dispersion. The slow nature of ξ and τ is expressed through one small parameter ϵ, the smallness of which is adequately defined in terms of some physical parameters. The parameter ϵ is also used to carry out the perturbative procedure leading to the model equation.

Nonlinear dynamics of short waves has been the subject of intensive study for a long time, most of which has been focussed on the evolution and modulation of *short wavetrains*. The modulation of weakly linear or nonlinear short waves riding on long waves were considered by Longuet-Higgins and Stewart [15], Schwartz [16], Hogan [17], Longuet-Higgins [18] and Zhang and Melville [19]. For a full account on the modulation of short wavetrains on water of intermediate or great depth please see the excellent book by Mei [20]. Another already classical line of investigation in short-wave dynamics is the theory of nonlinear interaction between a long wave and a train of short waves (Benney's theory). The earliest studies along this line were made by Nishikawa *et al* [21], Benney [22], Kawahara and Jeffrey [23], Woodruff and Messiter [24], and recently by Ramamonjiarisoa [25].

In contrast, and beyond the context of waves in fluids, *nonlinear asymptotic dynamics of monochromatic short waves* has hardly been studied and only a few results are known.

The first steps towards the establishment of a systematic multiple-scales theory of (monochromatic) short waves in nonlinear and dispersive systems were given in [26–28]. There are two main problems associated in carrying out such studies: the introduction of asymptotic variables useful to the description of the nonlinear dynamics of short waves and the identification of nonlinear and dispersive models allowing such propagation.

This paper has two main purposes: *to establish the necessary conditions for the existence of short-wave propagation in dispersive systems* and *to study the dynamics of nonlinear short surface waves*.

(a) We begin with the introduction of the equivalent of the Gardner–Morikawa transformation in the case of short waves. This is done in section 2 from some basic considerations of wave dynamics. In the same section we introduce the conditions satisfied by the linear dispersion relation in the case of a system that propagates short waves.
(b) Based on the conditions given in (a), for the existence and linear propagation of short waves, we consider the problem of nonlinear short surface waves in an ideal fluid. This is done through the introduction and the study of several specific models. In section 3 we deduce a modified Green–Nagdhi (GN) system of equations which admits short waves. They are governed by a new nonlinear model equation for which explicit solutions are exhibited. In section 4 we deduce a GN system with surface tension and derive another

new model equation governing short waves. Section 5 is devoted to some known nonlinear models which allow us to introduce the notion of a *second asymptotic limit*. In section 6 we study short waves in the CH and one of the Boussinesq systems. Finally, section 7 is devoted to some conclusions and comments. In the appendix we sum up the results concerning the BBMP equation, its short-wave dynamics and the associated multiple time analysis.

2. Basic existence conditions for short-wave asymptotic dynamics

Let us consider a one-dimensional nonlinear and dispersive system. In the simplest case it will be represented by a scalar nonlinear evolution equation in a field $u(x, t)$ with a linear and dispersive part and with x and t, respectively, being space and time variables. This evolution is written as

$$\mathcal{L}(u_t, u_x, u_{xt}, \ldots) = \mathcal{N}(u, u_x, u_{xx}, \ldots) \tag{1}$$

where $\mathcal{L}$ contains only the linear terms in $u_t, u_x, u_{xt}, \ldots$ (subscripts denote partial derivatives). They represent evolution in t and linear dispersion of arbitrary degrees. The term $\mathcal{N}$ contains all the nonlinearities. The simplest solution of the linear part $\mathcal{L}$ is a plane progressive wave of arbitrary amplitude A given by

$$u(x, t) = A \exp \mathrm{i}(kx - \omega t) \tag{2}$$

where k is the wavenumber related to the wavelength l by $k = 2\pi/l$ and ω is the frequency which satisfies a dispersion relation of the form

$$\omega = W(k). \tag{3}$$

In (3) W may be a function of other parameters present in the system. To simplify we consider $W(k)$ as only a function of k and we also define k as $k = 1/l$ absorbing the 2π factor in l without loss of generality. The wavenumber k is arbitrary in (2) hence, *a priori*, all wavelengths are allowed as solutions of the linear system.

A short wave now is a wave with $l \to 0$, so from a geometrical point of view it is a *nearly local phenomenon* and we are looking for its *large time behaviour*. Therefore we need to introduce two appropriate variables: one space variable ζ describing a local pattern and a time variable τ measuring a long period of time. These are worked out by means of the definition

$$\zeta = \frac{x}{\epsilon} \tag{4}$$

$$\tau = \epsilon t \tag{5}$$

where ϵ is a small positive parameter. Thereby ζ and τ are of order one when x is very small and t is very large, appropriate for describing short waves asymptotically in time. The definitions (4) and (5) are meaningful if they are compatible with the plane progressive short-wave solution of $\mathcal{L}$. So, if the small parameter ϵ is related to the size of the wavelength $l = 1/k = \epsilon$, the definitions (4) and (5) follow from (2) if

$$W(k) \sim \frac{1}{k} \qquad k \to \infty. \tag{6}$$

As a matter of fact the more complete expression for $W(k)$ leading to the more general expressions for ζ and τ is a Laurent series with a simple pole at $k \to \infty$, so

$$W(k) = ak + \frac{b}{k} + \frac{c}{k^3} + \cdots \tag{7}$$

where $a, b, c \ldots$ are real. This expression for $W(k)$ yields, at order one in ϵ, to asymptotic variables slightly more general than (4) and (5), namely

$$\zeta = \frac{1}{\epsilon}(x - at) \tag{8}$$

$$\tau = \epsilon t. \tag{9}$$

Hence a dispersion relation $W(k)$ which is bounded for $k \to \infty$ or which has almost a simple pole in this limit yields an accurate definition of the *fast variable* ζ and the *slow variable* τ. We consider $W(k)$ to be real because there is no dissipation and $W(k)$ is odd in k because the medium is isotropic.

Expression (7) actually defines an infinite number of slow variables $\tau_1 = \tau, \tau_3, \tau_5, \ldots,$ that we are not going to consider in detail in this paper but which must be considered when treating the problem of secularities [26] (see the appendix). $W(k)$ in the form (7) defines a phase velocity $W(k)/k = v_p$ and a group velocity $\frac{\partial W(k)}{\partial k} = v_g$ finite in the short-wave limit

$$\lim_{k\to\infty} \frac{W(k)}{k} = \lim_{k\to\infty} \frac{\partial W(k)}{\partial k} = a + \mathcal{O}(1/k^2). \tag{10}$$

In (7) ($a \neq 0$) the conditions in (10) guarantee that the geometrical characteristic and the energy linearly propagate at the same velocity a at order ϵ. In (6) ($a = 0, b \neq 0$) v_p and v_g are still finite but opposite in sign and consequently the wave energy and the wave's geometrical behaviour propagate linearly in opposite directions.

The change of variables (8) and (9) and the small parameter ϵ provide us with the right tools to broach the study of the complete nonlinear problem. First all the derivatives in relation to x or t in $\mathcal{L}$ and in $\mathcal{N}$ are written using (8) and (9) as derivatives in ζ and in τ. Second an appropriate perturbative series in ϵ and a scaling of $u(x, t)$ is done in the form

$$u = \epsilon^n(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots) \qquad n = 1, 2, \ldots. \tag{11}$$

Finally a perturbative procedure in ϵ allows us to identify the nonlinear dynamics as a nonlinear model equation for u_0 in variables ζ and τ. The asymptotic nonlinear model equation conserves its form in the laboratory coordinates u, x and t if n in (11) is conveniently chosen.

3. Modified Green–Nagdhi equations

In this section we want to find under what conditions short surface waves exist in an ideal fluid, and what evolution equation (if any) governs its propagation. As established previously the initial system must have a linear dispersion relation satisfying for $k \to \infty$ the asymptotic behaviour given by (6) or (7). This is a *sine qua non* condition which allows us to introduce the variables (4), (5) or (8), (9).

As a consequence the classical theory of the one-dimensional surface gravity waves on water of arbitrary and uniform depth h does not work. The associated linear dispersion relation

$$W(k) = [gk \tanh(kh)]^{\frac{1}{2}} \tag{12}$$

where g is the acceleration of gravity [29], cannot be developed in the form of (7).

Dispersion relations are wholly determined by the conditions under which the system is linearized. Hence we are facing the following problem: is there an alternative linearization of the surface water wave problem conducive to the adequate asymptotic variables ζ and τ? The answer is yes and the price to pay is abandoning the classical conditions of linearization.

To establish the dispersion relation (12) the physical hypothesis of the hydrostatic pressure condition [30, 31] is used, so as to eliminate the pressure in a trivial way. Also the initial state of the fluid is considered at rest or as an uniform stream [32]. Therefore we should search for modifications of both, *initial hydrodynamics conditions* and *physical hypothesis*.

The appropriate initial hydrodynamics state consists of an *initial surface flow*. We find that nonlinear short surface wave builds up as a result of the superposition of two motions: an oscillatory flow and a initial laminar flow of the surface. The oscillatory flow corresponds to mechanical perturbations which propagate like a wave. The initial laminar surface flow may be created by the action of an external wind. A surface wind on a lake which produces surface flow is the physical environment idealized by the above mentioned phenomenon [33, 34].

Therein we are going to study perturbations to a ideal fluid which is ideal, i.e. inviscid, incompressible, homogeneous with density σ and without surface tension. An initial surface displacement is then assumed *'ab initio'*.

Let the particles of this continuum medium be identified by a fixed rectangular Cartesian system of centre O and axes (x, y, z) with Oz being the upward vertical direction. We assume symmetry in y and we will only consider a sheet of fluid in the xz plane. This fluid sheet is moving in a domain with a rigid bottom at $z = 0$ and an upper free surface at $z = \phi(x, t)$. The vector velocity is $\vec{v} = (u, w)$. Thanks to the homogeneity and the incompressibility the continuity equation reduces to

$$\vec{\nabla} \cdot \vec{v} = u_x(x, z, t) + w_z(x, z, t) = 0 \qquad 0 \leqslant z \leqslant \phi(x, t) \tag{13}$$

where $\vec{\nabla} = (\partial_x, \partial_z)$. The Euler equations of motions (momentun conservation) of a fluid under gravity g and for $0 \leqslant z \leqslant \phi(x, t)$ are

$$\sigma \dot{u}(x, z, t) = -p^*_x(x, z, t) \tag{14}$$

$$\sigma \dot{w}(x, z, t) = -p^*_z(x, z, t) - g\sigma \tag{15}$$

where $p^*(x, z, t)$ is the pressure and an overdot denotes the material derivative.

We now complete the fundamental equations of continuity and momentun conservation with appropriate kinematic and dynamic boundary conditions. Let $S(x, z, t)$ be the interface between the inviscid fluid sheet and the external medium (air). We represent $S(x, z, t)$ by the (classical) equation

$$S(x, z, t) = \phi(x, t) - z = 0. \tag{16}$$

The kinematic condition is that the normal velocity of the surface $S(x, z, t)$ must equal the velocity of the fluid sheet normal to the surface. The normal velocity of the surface is [34]

$$-\frac{S_t}{\|\vec{\nabla} S\|} \tag{17}$$

while the velocity on the surface $z = \phi(x, t)$ is

$$\vec{v} = (u + c_0, w) \qquad z = \phi(x, t) \tag{18}$$

whose normal component is

$$\vec{v} \cdot \frac{\vec{\nabla} S}{\|\vec{\nabla} S\|}. \tag{19}$$

From (17), (18) and (19) the kinematic conditions read

$$\phi_t + u\phi_x - w + c_0\phi_x = 0 \qquad z = \phi. \tag{20}$$

Equation (18) leading to (20) lies at the heart of our approach. It points out that an external agent—wind in our case—drives the particles belonging to $S(x, z, t)$. The motion is uniform, of value c_0 and in the x-direction only. So $S(x, z, t)$ is a surface of discontinuity for the u component of $\vec{v}$, which experiences finite jumps of value c_0 in $z = \phi$.

At the surface of the sheet $z = \phi$, there is a constant normal pressure p_0. At the bed $z = 0$, there is an unknown pressure $p^*(x, 0, t)$ and the normal fluid velocity is zero: $w = 0$.

Let us now introduce the necessary physical hypothesis in order to obtain a suitable $W(k)$. Several types of physical approximations can be used to this end. Here we adopt an approximation in the velocity field used by Green *et al* in [35–37] and based on the monumental work of Naghdi [38]. We assume that u is independent of z: $u = u(x, t)$. This is equivalent to considering the vertical component w to be a linear function of z. This simple and realistic assumption, called *columnar hypothesis* [39], enables us to satisfy exactly the equation of incompressibility and the boundary conditions at the bed. Hence $u = u(x, t)$ and from (13) we have

$$w = z\xi(x, t) \qquad \xi(x, t) = -u_x(x, t). \tag{21}$$

Now we integrate (14) for the variable z from $z = 0$ to ϕ (the integration is granted by Riemann's condition of integrability). The result is

$$\sigma(u_t + uu_x)\phi = -p_x \tag{22}$$

where we use $\dot{u} = u_t + uu_x$ and

$$p(x, t) = \int_0^{\phi(x,t)} p^*(x, z, t)\,\mathrm{d}z - p_0\phi(x, t). \tag{23}$$

Next, we multiply equation (15) by z and integrate from $z = 0$ to ϕ which yields

$$\sigma(\xi^2 + \dot{\xi})\frac{\phi^3}{3} + \sigma g\frac{\phi^2}{2} = p. \tag{24}$$

Note that we do not use the hydrostatic pressure condition in order to eliminate p. The pressure p is eliminated using (24) in (22) and, with the help of (21) we eventually obtain

$$u_t + uu_x + g\phi_x = \phi\phi_x(u_{xt} + uu_{xx} - (u_x)^2) + \frac{\phi^2}{3}(u_{xt} + uu_{xx} - (u_x)^2)_x. \tag{25}$$

The remaining upper boundary condition on ϕ reads using (21)

$$\phi_t + (\phi u)_x + c_0\phi_x = 0. \tag{26}$$

With $c_0 = 0$, the system (25), (26) is the GN system of equations [35]. The inclusion of the term $c_0\phi_x$ drastically changes its dynamics because it cannot be eliminated either by a Galilean transformation or by a rescaling of u or ϕ. These extended GN equations represent the nonlinear interaction between two separate forms of motion: a wave motion associated with the elastic response of the fluid to a perturbation and a surface (uniform) motion generated by an external agent.

The linear dispersion relation $W(k)$ of the modified GN system is obtained assuming

$$\phi(x, t) = h + \exp \mathrm{i}[(kx - W(k)t)] \tag{27}$$

$$u(x, t) = \exp \mathrm{i}[(kx - W(k)t)] \tag{28}$$

where h is the unperturbed depth of the fluid. So $W(k)$ satisfies the second-order polynomial

$$W^2(k) - W(k)c_0k - \frac{3k^2hg}{3 + k^2h^2} = 0. \tag{29}$$

One of the two solutions of (29) has the asymptotic short-wave behaviour given by

$$W(k) = \frac{-3g}{c_0h}\left(\frac{1}{k}\right) + \frac{3}{c_0h^2}\left(\frac{3g}{h} + \frac{3g^2}{c_0^2}\right)\left(\frac{1}{k^3}\right) + \mathcal{O}\left(\frac{1}{k^5}\right) \tag{30}$$

which is of the form (6). Note that for $c_0 = 0$ (29) gives a $W(k)$ which behaves well only for long waves ($k \to 0$). Thereby we define short-wave variables of type (4) and (5) which define the operators

$$\frac{\partial}{\partial x} = \frac{1}{\epsilon}\frac{\partial}{\partial \zeta} \tag{31}$$

and

$$\frac{\partial}{\partial t} = \epsilon \frac{\partial}{\partial \tau}. \tag{32}$$

The nonlinear dynamics of short waves of small amplitude, in relation to h, is found by means of the expansions

$$\phi = h + \epsilon^2(H_0 + \epsilon^2 H_2 + \epsilon^4 H_4 + \cdots) \tag{33}$$

$$u = \epsilon^2(U_0 + \epsilon^2 U_2 + \epsilon^4 U_4 + \cdots) \tag{34}$$

where H_i and U_i ($i = 0, 2, \ldots$) are functions of ζ and τ. Substituting (33) and (34) in (25) and (26) and using (31) and (32) we obtain at a lower order in ϵ

$$h(U_0)_\zeta + c_0(H_0)_\zeta = 0 \tag{35}$$

$$g(H_0)_\zeta = \frac{h^2}{3}\{(U_0)_{\tau\zeta} + U_0(U_0)_{\zeta\zeta} - (U_0)_\zeta^2\}_\zeta. \tag{36}$$

Using (35) in (36) and going back to the laboratory fields and co-ordinates ϕ, u, x, and t and integrating once we eventually arrive at the equation

$$u_{xt} = -\frac{3g}{hc_0}u - uu_{xx} + (u_x)^2. \tag{37}$$

This is a model equation governing the asymptotic dynamics of short surface wind waves. It contains a linear dispersive term and two nonlinear terms of opposite signs.

Equation (37) has several types of interesting solutions. The peakon solution is

$$u(x,t) = -\alpha\lambda^2 \exp\left(-\left|\frac{x + \alpha\lambda^2 t}{\lambda}\right|\right) \qquad \alpha = -\frac{3g}{hc_0} \tag{38}$$

where the width λ is a free parameter. Unlike the peakon of the CH equation the amplitude $(-\alpha\lambda^2)$, the velocity $(\alpha\lambda^2)$ and the width (λ) are related.

The static compacton solution is

$$u = -8\alpha\lambda^2 \cos^2\left(\frac{x}{4\lambda}\right) \qquad \left|\frac{x}{\lambda}\right| \leqslant 2\pi \tag{39}$$

and $u = 0$ otherwise. Unlike the compacton solution recently introduced and investigated by Rosenau and Hyman [40, 41], this solution presents a dependence between the width and the amplitude. Despite the fact that this solution is unnatural in the physics under consideration, it shows that (37) is an adequate mathematical tool for modelling stationary short patterns in nature.

Solutions (38) and (39) are coherent structures analogous to the solitary wave of KdV. Moreover, contrary to KdV, (37) possess a plane monochromatic wave solution of arbitrary amplitude $A \exp i(kx - \Omega t)$ with the dispersion relation $\Omega(k) = -3g/khc_0$ identical to that of the linearized system[1]. Moreover this dispersion relation is a function of k which behaves well in the short-wave limit. We have shown in [42] that this plane wave solution is Benjamin–Feir unstable.

[1] A plane wave solution of a nonlinear and dispersive equation occurs also for the nonlinear Schrödinger equation: $\psi_t + \psi_{xx} + |\psi|^2\psi = 0$ but there Ω is A-dependent.

The underlying mechanism responsible for the structural stability of solutions of the CH equation or the KdV equation with nonlinear dispersion is the balance between nonlinear dispersion, nonlinear convection and nonlinearity. Indeed these equations are nonlinear evolution equations without linear dispersion (the plane wave is not a solution of the linear associated evolution equations). However, the structural stability of (38), (39) comes from the balance between the linear dispersion and nonlinear terms as for the classical solitary waves solutions of long-wave models.

Contrary to CH or KdV with nonlinear dispersion, equation (37) is Galilean invariant.

4. The Green–Nagdhi system with surface tension

In this section we study the role of surface tension in short-wave nonlinear dynamics. That is we will consider surface waves originating from two restoring forces: the gravitational force and the surface tension. Note that again in this case the surface is activated by two agents. The classical capillary waves theory [32] gives as a dispersion relation the expression

$$W(k) = \left[k\left(g + \frac{Tk^2}{\sigma}\right)\tanh(kh)\right]^{\frac{1}{2}} \tag{40}$$

where T is the surface tension. $W(k)$ does not satisfy the necessary behaviour for $k \to \infty$ and we cannot define the variables ζ and τ. As for (12), the dispersion relation (40) is obtained from an initial static state of the fluid and under hydrostatic pressure conditions. We leave as the initial state the static one but we remove the hypothesis of hydrostatic pressure as in the previous case.

The basics equations for $0 \leqslant z \leqslant \phi(x, t)$ with the surface acted on by gravity and surface tension are

$$u_x(x, z, t) + w_z(x, z, t) = 0 \tag{41}$$

$$\sigma\dot{u}(x, z, t) = -p_x^*(x, z, t) \tag{42}$$

$$\sigma\dot{w}(x, z, t) = -p_z^*(x, z, t) - g\sigma \tag{43}$$

with the boundary conditions given by

$$w = 0 \qquad z = 0 \tag{44}$$

$$\phi_t + u\phi_x - w = 0 \qquad z = \phi \tag{45}$$

$$p^* = p_0 - \frac{T\phi_{xx}}{(1 + \phi_x^2)^{\frac{3}{2}}} \qquad z = \phi \tag{46}$$

where all quantities have the same meaning as in section 3. The reduction is obtained by: (1) integration of (42) in the variable z from $z = 0$ to $\phi(x, t)$; (2) multiplication of (43) by z and integration in the variable z from $z = 0$ to $\phi(x, t)$; (3) use of (41) in (45) and the hypothesis of independence of u in z. As a result we get the system

$$\phi_t + (u\phi)_x = 0 \tag{47}$$

$$\sigma\phi(u_t + uu_x) = -p_x + T[(1 + \phi_x^2)^{-\frac{1}{2}}]_x \tag{48}$$

$$\sigma\frac{\phi^3}{3}(-u_{xt} - uu_{xx} + u_x^2) = p + T\phi\phi_{xx}[1 + \phi_x^2]^{-\frac{3}{2}} - \frac{g\sigma\phi^2}{2} \tag{49}$$

where $p(x, t)$ is the same functions as those defined in (23). The system (47), (48) and (49) apart from some multiplicative constants was derived firstly by Green *et al* in [35] from the theory of directed fluid sheets [38] which is a method that is rather different from that employed here.

Let us first of all analyse the linear associated system. We assume that

$$\phi(x,t) = h + v(x,t). \tag{50}$$

Susbtituting in (47), (48) and (49) and retaining only the linear terms in u and v we obtain

$$v_t + hu_x = 0 \tag{51}$$

$$\sigma h u_t = -p_x \tag{52}$$

$$-\frac{\sigma h^3}{3} u_{xt} = p + Thv_{xx} - \frac{gh}{2}(h^2 + 2hv). \tag{53}$$

This system can be reduced to only one equation in u which reads

$$u_{tt} + \frac{Th}{\sigma} u_{xxxx} - \frac{h^3}{3} u_{xxtt} - ghu_{xx} = 0. \tag{54}$$

The dispersion relation corresponding to (54) is

$$W(k) = k \left(\frac{gh + \frac{Th}{\sigma}k^2}{1 + \frac{h^2k^2}{3}} \right)^{\frac{1}{2}} \tag{55}$$

which for $k \to \infty$ has the following behaviour:

$$W(k) = \left(\frac{3T}{h\sigma}\right)^{\frac{1}{2}} \left[k + \left(\frac{g\sigma}{4T} - \frac{3}{2h^2} \right) \frac{1}{k} + \mathcal{O}\left(\frac{1}{k^3}\right) \right]. \tag{56}$$

$W(k)$ has the asymptotic form (7) and enables us to define the variables ζ and τ as

$$\zeta = \frac{1}{\epsilon}(x - v_f t) \tag{57}$$

$$\tau = \epsilon t \tag{58}$$

leading to (until order ϵ)

$$\frac{\partial}{\partial x} = \frac{1}{\epsilon}\frac{\partial}{\partial \zeta} \tag{59}$$

$$\frac{\partial}{\partial t} = -\left(\frac{v_f}{\epsilon}\right)\frac{\partial}{\partial \zeta} + \epsilon \frac{\partial}{\partial \tau}. \tag{60}$$

Now, we can look for nonlinear dynamics of short waves in (47)–(49) with the help of (59) and (60) and with a rescaling of u, ϕ and p given by

$$u = \epsilon^2 (u_0 + \epsilon^2 u_2 + \cdots) \tag{61}$$

$$\phi = h + \epsilon^2 (w_0 + \epsilon^2 w_2 + \cdots) \tag{62}$$

$$p = p_0 + \epsilon^2 p_2 + \cdots \tag{63}$$

where u_i, w_i, and p_i $(i = 0, 2, \ldots)$ are functions of ζ and τ. The order ϵ^{-1} in (47) and (48) and the order zero in (49) give the system

$$hu_{0,\zeta} - v_f w_{0,\zeta} = 0 \tag{64}$$

$$p_{0,\zeta} = 0 \tag{65}$$

$$\frac{\sigma h^3}{3} v_f u_{0,\zeta\zeta} - Thw_{0,\zeta\zeta} = p_0 - \frac{g\sigma h^2}{2}. \tag{66}$$

Equation (65) gives p_0 to be a constant. Derivating (66) once and (64) twice in relation to ζ we obtain a linear homogeneous system for $u_{0,\zeta\zeta\zeta}$ and $w_{0,\zeta\zeta\zeta}$. This has a nontrivial solution if and only if its determinant is zero. This condition determines v_f as

$$v_f^2 = \frac{3T}{\sigma h}. \tag{67}$$

Next we obtain using (64) and (66)

$$u_0 = \frac{v_f}{h} w_0 \tag{68}$$

$$p_0 = \frac{g\sigma h^2}{2} \tag{69}$$

where we had disregarded a constant of integration. The next order gives the system

$$hu_{2,\zeta} - v_f w_{2,\zeta} = -w_{0,\tau} - (u_0 w_0)_\zeta \tag{70}$$

$$p_{2,\zeta} = v_f \sigma h u_{0,\zeta} - T w_{0,\zeta} w_{0,\zeta\zeta} \tag{71}$$

$$\frac{\sigma h^3}{3} v_f u_{2,\zeta\zeta} - Thw_{2,\zeta\zeta} = p_2 - g\sigma h w_0 - \tfrac{3}{2} Thw_{0,\zeta\zeta} w_{0,\zeta}^2$$
$$+Thw_{0,\zeta\zeta} w_0 \frac{\sigma h^3}{3}(u_{0,\zeta\tau} + u_0 u_{0,\zeta\zeta} - u_{0,\zeta}^2) - \sigma h^2 v_f w_0 u_{0,\zeta\zeta}. \tag{72}$$

Equation (71) gives (using (68))

$$p_2 = v_f \sigma h u_0 - \frac{Th^2}{2v_f^2} u_{0\zeta}^2. \tag{73}$$

Substituting p_2 in (70) and in (72) and derivating (70) once in ζ we obtain a linear nonhomogeneous system of equations for $u_{2,\zeta\zeta}$ and $w_{2,\zeta\zeta}$. The determinant of the associated homogeneous system is zero owing to (67) and therefore, we will have a solution if the Fredholm solvability condition is satisfied. This condition gives a nonlinear equation for u_0 and reads

$$u_{0,\zeta\tau} = \frac{3g}{2v_f h}(1-3\theta)u_0 + \frac{3h^2}{4v_f} u_{0,\zeta\zeta} u_{0,\zeta}^2 - \frac{1}{4} u_{0,\zeta}^2 - \frac{1}{2} u_0 u_{0,\zeta\zeta} \tag{74}$$

where θ is a dimensionless parameter known as the Bond number

$$\theta = \frac{T}{\sigma h^2 g}. \tag{75}$$

Equation (74) represents the nonlinear dynamics of short waves of order ϵ^2 generated by gravity and tension forces. In the laboratory the coordinates (74) are

$$u_{xt} + v_f u_{xx} = \frac{3g}{2v_f h}(1-3\theta)u + \frac{3h^2}{4v_f} u_{xx} u_x^2 - \frac{1}{4} u_x^2 - \frac{1}{2} u u_{xx}. \tag{76}$$

The sign of the important linear term in u is governed by θ. For $\theta = 1/3$ there is an important change of regime because the linear term changes its sign. The critical value $\theta = 1/3$ is known in the literature and has motivated some classical debates [43, 44]. We leave for a future analytical and/or numerical work the study of solutions of (76) as a function of θ. We also leave for a future work the study of other asymptotic model equations associated with different scalings of u, ϕ and p.

5. Model equations and second asymptotic limits: a classification

From the basic hydrodynamics equations governing surface gravity waves, several long-wave and shallow-water models can be derived. Common procedures, different from the multiscale expansion method used here, are perturbative expansions in two small parameters. One of them measures amplitude in relation to depth and the other measures depth to wavelength ratio. We can consider these model equations as the result of a *first asymptotic limit* and thus we obtained the universally known KdV and one of the system of Boussinesq type (B), which in a dimensional formulation [32] are

$$\eta_t + c_0\left(1 + \frac{3}{2}\frac{\eta}{H_0}\right)\eta_x + \gamma\eta_{xxx} = 0 \tag{77}$$

and

$$\begin{aligned} &H_t + (Hu)_x = 0 \\ &u_t + uu_x + gH_x + \tfrac{1}{3}c_0^2 H_0 H_{xxx} = 0 \end{aligned} \tag{78}$$

where H_0 is the unperturbed depth of the fluid, $c_0 = (gH_0)^{1/2}$, $\gamma = (1/6)c_0 H_0^2$, $u(x,t)$ the x component of the velocity and $\eta(x,t)$ a perturbation of the surface ($H = H_0 + \eta(x,t)$). These equations are the lowest order in the perturbative procedure which retains nonlinearity and dispersion. The zeroth order leads to the linear and nondispersive wave equation given by

$$\eta_{tt} - c_0^2\eta_{xx} = 0 \tag{79}$$

or in terms of H

$$H_{tt} - c_0^2 H_{xx} = 0. \tag{80}$$

If we use the zero-order results of (79) or (80) in the linear and dispersive term of KdV or B, which are first order, we obtain instead of (77) or (78) the equations

$$\eta_t + c_0\left(1 + \frac{3}{2}\frac{\eta}{H_0}\right)\eta_x + \frac{\gamma}{c_0}\eta_{xxt} = 0 \tag{81}$$

and

$$\begin{aligned} &H_t + (Hu)_x = 0 \\ &u_t + uu_x + gH_x + \tfrac{1}{3}H_0 H_{xtt} = 0. \end{aligned} \tag{82}$$

Equation (81) is the BBMP equation introduced in [7] and system (82) (B*) is the one favoured by Boussinesq in [1]. Another interesting model equation with a status analogous to the above models is the recently introduced CH equation. It was derived by Camassa and Holm in [8] from shallow-water theory, by Fokas and Fuchssteiner in [45], and by Fokas in [46] and reads

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{83}$$

where κ is a constant related to the shallow water wave speed c_0 and u is the fluid velocity in the x-direction.

Based on these models and in its associated linear dispersion relation we can introduce the notion of a *second asymptotic limit: a model equation allows a second asymptotic limit if by use of reductive perturbation methods we can obtain from it another dispersive, linear or nonlinear model equation*.

It is clear that the necessary asymptotic variables to build up the second asymptotic limit will be dictated by the linear dispersion relation of the initial model.

Let us explain this position by considering the linear dispersion relation of KdV, CH, B, BBMP and B* and the allowed asymptotic limits. We start by linearizing KdV and CH around the zero-solution and B around a constant solution H_0 for H ($H = H_0 + \eta$) and u around zero. We have (where W_{KdV} means the linear dispersion relations of KdV, etc)

$$W_{\mathrm{KdV}}(k) = c_0 k - \gamma k^3 \tag{84}$$

$$W_{\mathrm{CH}}(k) = \frac{2\kappa k}{1 + k^2} \tag{85}$$

$$W_{\mathrm{B}}(k) = c_0 k\left(1 - \frac{H_0^2 k^2}{3}\right)^{1/2}. \tag{86}$$

In linearizing BBMP or B* we must take into account that in their derivation (from (77) and (78)) we have used the linear and nondispersive equations $\eta_t = c_0\eta_x$ and $H_{tt} = c_0^2 H_{xx}$. Hence, the u-terms corresponding to velocities in BBMP and B* must be linearized around $c_0 = (gH)^{1/2}$. So, we have

$$W_{\mathrm{BBMP}}(k) = \frac{c_0^2 k}{c_0 - \gamma k^2} \tag{87}$$

$$W_{\mathrm{B}^*}(k) = \frac{6c_0 k}{3 + H_0^2 k^2}. \tag{88}$$

From $W_{\mathrm{KdV}}(k)$ and the associated v_{p} and v_{g} it is easy to see that the multiple-scale method can be applied for $k \to 0$. Therefore, a second long-wave limit of KdV is allowed but it gives the linear dispersive equation $\eta_t \sim \eta_{xxx}$. On the other hand, a second long-wave limit of B is possible leading to KdV. Subsequently, on account of this fact, we can say that KdV is a *hard asymptotic long-wave model*, because no nonlinear and dispersive model can be obtained from it by a limit process and B is a *weak asymptotic long-wave model* because it has KdV as the limit equation.

CH, BBMP, and B* are also weak asymptotic long-wave models because they allow a second long-wave asymptotic limit leading to different forms of the ubiquitous KdV.

We are now at the stage to formulate another issue concerning these models: *is a second asymptotic short-wave limit ($k \to \infty$) possible?*

Such a question is relevant for many reasons. First, when we use these model equations to model real physical systems we are often led to use them beyond the precise range of validity under which they were derived. In these situations we will very likely be in a short-wave region. Second, whenever we realise numerical discretizations of these models, short waves are introduced as secondary effects coming from finite difference methods and truncations. Third, initial profile solutions (wave-paquets), in terms of the Fourier integral, contain short-wave components which will have an effect on the dynamics involved.

It is easy to see that a short-wave limit ($k \to \infty$) is not possible in KdV, because $W_{\mathrm{KdV}}(k)$ is not of the form (6) or (7), and leads to v_f and v_{g} diverging. In linear B, we have $W_{\mathrm{B}}(k)$ and v_f and v_{g} go to imaginary functions for $k \to \infty$. Thus, linear KdV and linear B behave badly for the short-wave limit. Owing to the impossibility of defining the variables ζ and τ we cannot carry out either linear or nonlinear analysis here.

In contrast CH, BBMP, and B* have finite v_f and v_{g} values for $k \to \infty$ since the dispersion relations $W_{\mathrm{CH}}(k)$, $W_{\mathrm{BBMP}}(k)$, and $W_{\mathrm{B}^*}(k)$ are in the form of equations (6) or (7).

In addition we will see that nonlinear short-wave dynamics in CH, BBMP and B* leads to nontrivial asymptotic models. We can then complete our classification of model equations saying that CH, BBMP, and B* are *weak asymptotic long-short model equations admitting both long and short second asymptotic limits*. If this classification is applied to the modified GN equation and to the GN with surface tension we see that they are weak long-short models. Second short-wave asymptotic limits lead to models (37) and (76) and second long-wave asymptotic limits lead to KdV.

In the next section we will study short-wave dynamics in CH and B^*. Short waves in BBMP are given in [26]. For completeness we sum up these results in the appendix.

6. Short waves in the Camassa–Holm and in the Boussinesq equations

In [8] Camassa and Holm discuss equation (83) and its solutions in the particular case $\kappa = 0$. Although unphysical this limit is of great interest, because it leads to peakon solutions. As the CH equation is completely integrable (for all κ) N-peakon dynamics possess a behaviour analogous to the N-soliton solution of integrable nonlinear and dispersive systems. The solutions of CH for all κ were studied in [48]. From a dynamic point of view peakons of CH are the result of an equilibrium between nonlinearity and nonlinear dispersion, because linear dispersion goes to zero in the $\kappa \to 0$ limit, as we can see from its dispersion relation

$$W_{\mathrm{CH}}(k) = \frac{2\kappa k}{1+k^2}.$$

Furthermore, from $W_{\mathrm{CH}}(k)$ with $\kappa = 0$ neither a slow space variable nor a fast space variable can be defined and thus, the peakon solutions of CH are unusual objects which do not represent long waves or short waves in the usual sense. In fact, the amplitude, velocity and width of the peakon are not interrelated as in the solitary wave of KdV or in the peakon solution (38) of (37).

Let us now consider the case $\kappa \neq 0$ and look for short waves. From $W_{\mathrm{CH}}(k)$ we can define the variables ζ and τ of (4) and (5). A perturbative procedure with a scaling in u in the form $u = \epsilon^2(v_0 + \epsilon^2 v_2 + \cdots,)$ yields the equation

$$v_{0,\zeta\tau} = 2\kappa v_0 - \tfrac{1}{2}v_{0,\zeta}^2 - v_0 v_{0,\zeta\zeta} \tag{89}$$

or in the laboratory the fields and variables u, x and t

$$u_{xt} = 2\kappa u - \tfrac{1}{2}u_x^2 - u u_{xx}. \tag{90}$$

This equation is completely integrable and belongs to the Harry–Dym hierarchy [49, 50]. In the case $\kappa = 0$ it was studied in [51], where its bi-Hamiltonian structure and solutions were exhibited.

Finding an equation belonging to the Harry–Dym hierarchy as a short-wave asymptotic limit of a model equation is new and unexpected. It is coherent with the Calogero theory of the S-integrability [9] because CH is an integrable equation.

Let us now consider short waves in B*. The linear dispersion relation (88) allows us to define the variables ζ and τ of (4) and (5). Furthermore, we introduce the scalings

$$H = H_0 + \epsilon^2(V_0 + \epsilon^2 V_2 + \cdots) \tag{91}$$

$$u = c_0 + \epsilon^2(W_0 + \epsilon^2 W_2 + \cdots) \tag{92}$$

and substitute the developments (91) and (92) and the operators (31) and (32) in (82). We obtain at order ϵ^{-1} a system whose solution is given by

$$W_{0,\zeta} = -\frac{c_0}{H_0}V_{0,\zeta}. \tag{93}$$

At the next order we get

$$V_{0,\tau} + H_0 W_{2,\zeta} + c_0 V_{2,\zeta} + (V_0 W_0)_\zeta = 0 \tag{94}$$

$$W_{0,\tau} + c_0 W_{2,\zeta} + W_0 W_{0,\zeta} + g V_{2,\zeta} + \frac{H_0}{3}V_{0,\tau\tau\zeta} = 0. \tag{95}$$

This is an inhomogeneous linear system of equations for $W_{2,\zeta}$, $V_{2,\zeta}$ of determinant zero because $c_0^2 = gH_0$. Therefore, we will have a solution if the Fredholm solvability condition is satisfied. This condition gives a nonlinear equation for W_0 and reads

$$W_{0,\tau\tau\zeta} = \frac{3c_0}{H_0^2}(2W_{0,\tau} + 3W_0 W_{0,\zeta}) \tag{96}$$

which in the laboratory coordinates u, x, t is

$$u_{xxt} = \frac{3c_0}{H_0^2}(2u_t - 3c_0 u_x + 3u u_x). \tag{97}$$

The cnoidal wave solution of (97) is

$$u = c_0 + \frac{3c_0\kappa^2}{H_0^2 k^2 (1+\kappa)^2} sn^2 \left(kx - \frac{3c_0 t}{2kH_0^2(1+\kappa)^2} \right) \tag{98}$$

where $sn(x, \kappa)$ is the Jacobian elliptic function of modulus κ and k is the wavenumber. Owing to the fact that for $\kappa \to 1$ we have $sn^2(x) \to \tanh^2(x)$ from (98) we obtain the soliton solution of (97)

$$u = c_0 + \frac{3c_0}{4H_0^2 k^2} \tanh^2 \left(kx - \frac{3c_0 t}{4kH_0^2} \right). \tag{99}$$

Solutions (98) and (99) represent an equilibrium between linear dispersion and nonlinearity and have frequencies behaving well only in the short-wave limit $k \to \infty$.

7. Conclusion and final comments

We have shown that nonlinear evolution model equations that govern short surface waves result from nonlinear and dispersive hydrodynamics, when the free surface is activated by two factors at least. For the modified GN system, the Boussinesq system, the CH equation as well as the BBMP equations, gravitational effects are superposed to a uniform (pre-established) surface motion. In the case of GN with surface tension the two agents are the surface tension and the restored gravitational force. Static compacton and peakon solutions with interrelated amplitude, velocity and width parameters were seen. The very important role played by the balance between linear dispersion, and nonlinear terms (such as for long-wave dynamics) was shown in the explicit solutions and several new model equations governing asymptotic dynamics of short waves were derived. In particular in the case with surface tension, a new model equation was established which will enable us to study the dynamics as a function of a critical Bond number.

The classical connection between equations of the KdV hierarchy and long-wave dynamics [47] was made broader with a new issue: *the Harry–Dym hierarchy contains equations related to short-wave dynamics*.

Note that in all nonlinear short-wave models derived in this paper the linear limit is the equation

$$u_{xt} - au = 0 \tag{100}$$

where a is a constant. The progressive plane wave solution of this equation has its dispersion relation $W(k)$ in the form

$$W(k) = \frac{a}{k}. \tag{101}$$

From (101) we see that the simplest evolution equation associated with short waves is intrinsically dispersive because $v_{\mathrm{p}} = W(k)/k = a/k^2$. This fact radically differs from the case of long-wave dynamics, for which the simplest equation is the wave equation

$$u_t - au_x = 0 \tag{102}$$

which is nondispersive. Note also that equations (37), (76), (90), (97) and (105) (see appendix) are not evolution equations 'stricto sensu', but are integro-differential equations. The nonlocality of nonlinear evolution equations for short waves appears to be inherited from the basic nonlocality of (100) which can be written as

$$u_t - a \int_{-\infty}^{x} u \, \mathrm{d}x = 0. \tag{103}$$

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Appendix. Short waves and infinity time variables in the Benjamin–Bona–Mahony–Peregrine equation

Finally let us resume our investigation concerning short waves in the BBMP equation (81). BBMP was introduced with the purpose of avoiding the failure of KdV regarding short-wave dynamics (about this problem see [52]). Rescaling η, t and x it can be put in the simplest form as already studied in [26, 27]

$$u_t + u_x - u_{xxt} = 3(u^2)_x. \tag{104}$$

The short-wave limit of (104) is given by the equation [26]

$$u_{xt} - u + 3u^2 = 0. \tag{105}$$

Its solitary wave solution is

$$u = \frac{1}{2}\text{sech}^2\left(kx - \frac{t}{4k}\right) \tag{106}$$

representing the balance between nonlinearity and linear dispersion. Other explicit solutions, blow-up and a new nonlinear instability were studied in [53].

This equation provides a good example of formations of a soliton solution by the nonlinear interaction between all degrees of dispersion with time. Let us explain this point. The dispersion relation of BBMP (87) allows us to define an asymptotic fast space variable ζ, and infinite slow time variables $\tau_1, \tau_3, \tau_5, \ldots$. According to this we have the operators

$$\frac{\partial}{\partial x} = \frac{1}{\epsilon}\frac{\partial}{\partial \zeta} \qquad \frac{\partial}{\partial t} = \epsilon\frac{\partial}{\partial \tau_1} + \epsilon^3\frac{\partial}{\partial \tau_3} + \epsilon^5\frac{\partial}{\partial \tau_5}\cdots$$

which allows us to study the behaviour of a short wave for $t \to \infty$. Assuming the expansion

$$u = u_0 + \epsilon^2 u_2 + \epsilon^4 u_4 + \cdots$$

BBMP gives at order ϵ^{-1} the model equation

$$u_{0,\zeta\tau_1} = u_0 - 3u_0^2.$$

As we have shown in [26] by using

(a) the soliton solution (equation (106)),
(b) the slow variables $\tau_1, \tau_3, \tau_5, \ldots (t \to \infty)$,
(c) the nonsecularity requirements of the series solution,

we can calculate $u_2, u_4, \ldots$ and sum up the series for u. We obtain the soliton of BBMP (in its canonical form)

$$u = -\frac{2k^2}{1-4k^2}\text{sech}^2\left[k\left(x + \frac{t}{4k^2-1}\right)\right].$$

Consequently we can state that: *from a linear point of view BBMP does not propagate short waves but from a nonlinear point of view short waves do propagate nonlinearly in BBMP and build up soliton-like solutions as $t \to \infty$.*

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